Millimeter Laser Ranging to the Moon:
a comprehensive theoretical model for advanced data analysis

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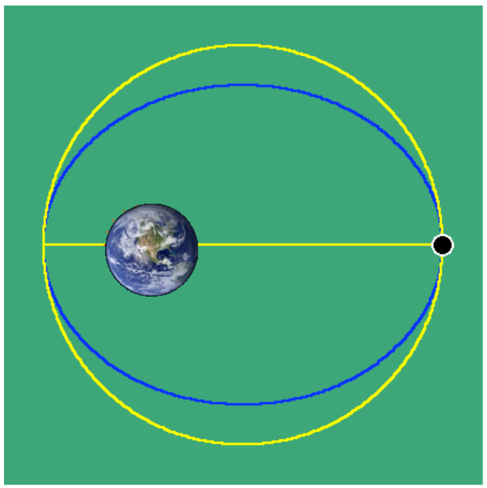
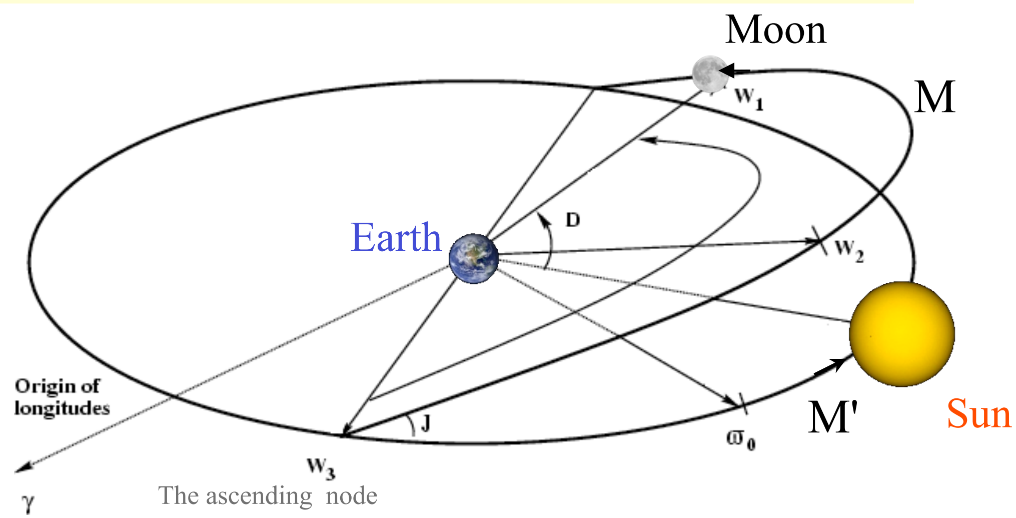
Next 25 slides:

- The basics of LLR
- Historical background
- The Newtonian Motion
- General Relativity at a glimpse
- PPN equations of motion
- Motivations behind PPN
- Gauge freedom in the lunar motion
- IAU 2000 theory of reference frames
- Lunar theory in a local-inertial reference frame
- Magnitude of synodic relativistic terms

No celestial body has required as much labor for the study of its motion as the Moon!

M – the mean anomaly of the Moon
M' – the mean anomaly of the Sun
D = M - M'





True longitude of the Moon = the mean longitude
(20905 km) + 377' *sin* M ECCENTRIC-1 (period 27.3 days)
+ 13' *sin* 2M ECCENTRIC-2 (period 13.7 days)
(3699 km) + 76' *sin* (2D - M) EVECTION (period 31.8 days)
(2956 km) + 39' *sin* 2D VARIATION (period 14.7 days)

Historical Background (before Einstein)

- **Newton** – the first theoretical explanation of the main lunar inequalities (1687)
- **Clairaut** – lunar theory with the precision of 1.5 arc-minute (1752)
- **Laplace** – the lunar theory with the precision of 0.5 arc-minute; secular acceleration; speed of gravity (1802)
- **Hansen** – the lunar theory and tables with the precision of 1 arc-second (1857)
- **Delaune** – an elliptic unperturbed orbit; 230 terms in the perturbing function; perturbation of the canonical set of elements; precision 1 arc-second (1860)
- **Hill** – rotating coordinates; Hill's equation; Hill's intermediate orbit; precision 0.1 arc-second (1878)
- **Brown** – extension of Hill's theory; Brown's tables; precision 0.01 arc-second (1919)

Historical Background (after Einstein)

- **De Sitter** – relativistic equations of the Moon; geodetic precession (1919)
- **Einstein-Infeld-Hoffmann** – relativistic equations of N-body problem; massive bodies as singularities of space-time (1938)
- **Fock-Petrova** – relativistic equations of N-body problem; massive bodies as extended fluid balls (1940)
- **Brumberg** – relativistic Hill-Brown theory of the Moon based on the EIH equations; eccentricity in relativistic term $e = 0$ (1958)
- **Baierlein** – extension of Brumberg's theory for $e \neq 0$ (1967)
- **Apollo 11** - LLR technique gets operational; ranging precision = a few meters (1969)
- **Nordtvedt** – testing the strong principle of equivalence with LLR (1972)
- **Standish** – JPL numerical ephemeris of the Moon and planets (DE/LE)
- **Brumberg-Kopeikin** – relativistic theory of reference frames in N-body problem; matching technique (1989)
- **Damour-Soffel-Xu** - relativistic theory of reference frames in N-body problem; relativistic multipole moments (1991)
- **IAU 2000** – relativistic resolutions on time scales and reference frames based on the BK-DSX papers
- **APOLLO** – new LLR technology at the Apache Point Observatory (2005); ranging precision 1 millimeter

Newtonian Equations of the Lunar Motion

$$\frac{d^2\vec{x}_L}{dt^2} = \underbrace{\vec{\nabla}U_E(\vec{x}_L) +}_{\text{Earth's gravity}} \underbrace{\vec{Q}_L}_{\text{Figure's effects}} + \vec{\nabla}\underbrace{\bar{U}(\vec{x}_L)}_{\text{Sun and planets}}$$

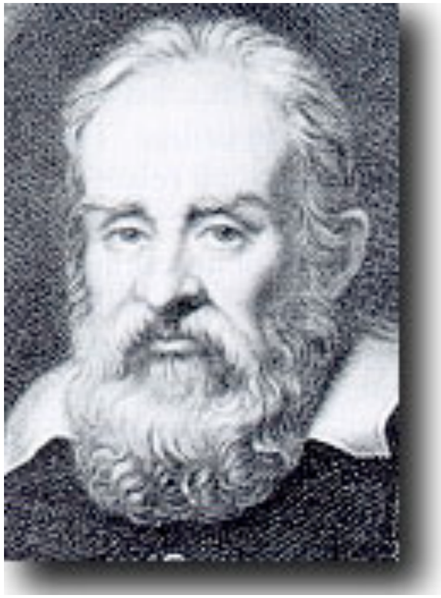
$$\frac{d^2\vec{x}_E}{dt^2} = \underbrace{\vec{\nabla}U_L(\vec{x}_E) +}_{\text{Moon's gravity}} \underbrace{\vec{Q}_E}_{\text{Figure's effects}} + \vec{\nabla}\underbrace{\bar{U}(\vec{x}_E)}_{\text{Sun and planets}}$$

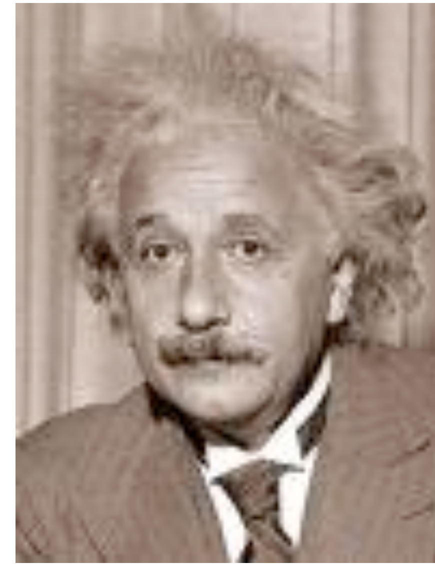
$$M = M_L + M_E; \qquad M\vec{x}_{\mathrm{cm}} = M_L\vec{x}_L + M_E\vec{x}_E; \qquad \vec{r} = \vec{x}_L - \vec{x}_E;$$

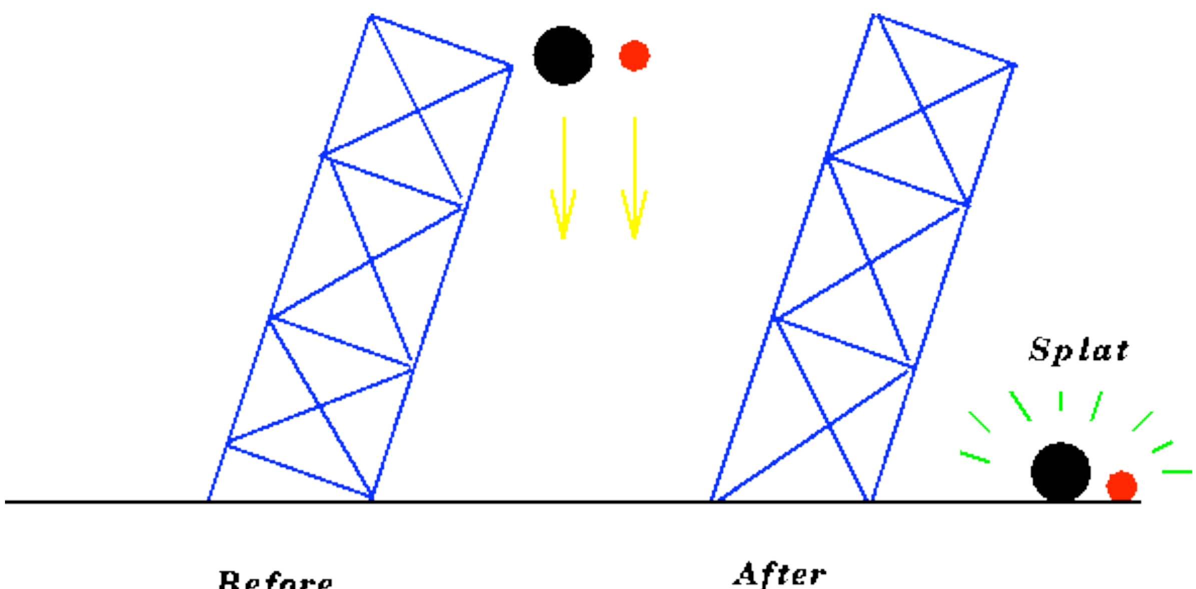
$$\frac{d^2\vec{x}_{\mathrm{cm}}}{dt^2} = \frac{M_L}{M}\vec{\nabla}\bar{U}(\vec{x}_L) + \frac{M_E}{M}\vec{\nabla}\bar{U}(\vec{x}_E) = \vec{\nabla}\bar{U}(\vec{x}_{\mathrm{cm}}) + (\text{tidal terms})$$

$$\frac{d^2\vec{r}}{dt^2} = \underbrace{\vec{\nabla}\left[U_E(\vec{x}_L) - U_L(\vec{x}_E)\right] +}_{\text{Earth-Moon gravity force}} \underbrace{\vec{\nabla}\left[\bar{U}(\vec{x}_L) - \bar{U}(\vec{x}_E)\right]}_{\substack{\text{Tidal gravity force from the Sun and planets.} \\ \text{Gradient of the perturbing potential.}}}$$

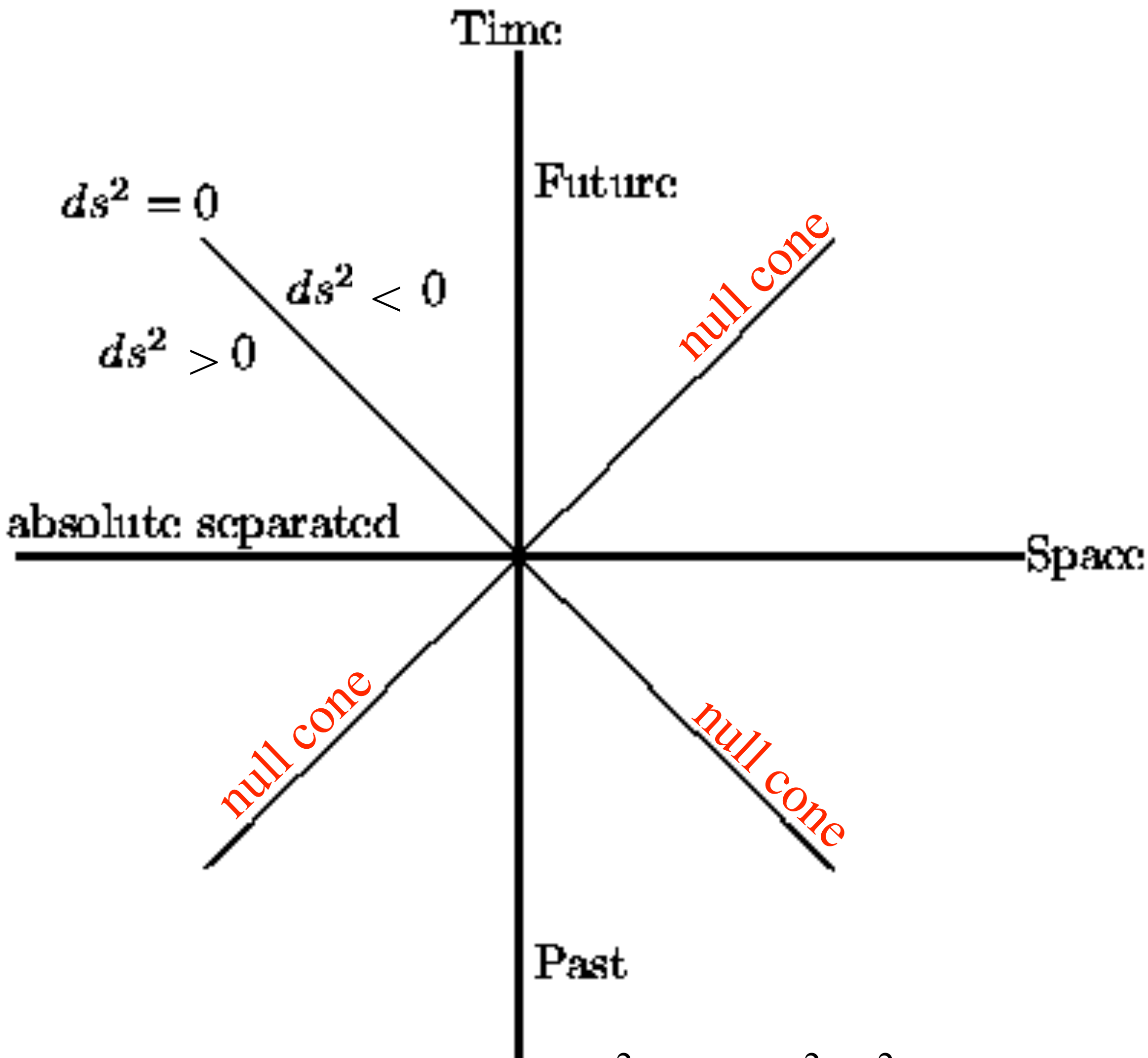
Gravitational Field is not a Scalar!







From Minkowski to Riemann geometry



$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \longrightarrow ds^2 = g_{00}c^2dt^2 + 2g_{0i}cdtdx^i + g_{ij}dx^idx^j$$

General Theory of Relativity at a glimpse

- The metric tensor ten gravitational potentials
- The affine connection the force of gravity

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(\frac{\partial g_{\beta\mu}}{\partial x^{\nu}} + \frac{\partial g_{\beta\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\beta}} \right)$$

- The Riemann tensor ≡ the relative (tidal) force of gravity
- The Principle of Equivalence the covariant derivative ∇
- The Gravity Field Equations

$$R^{\alpha}{}_{\beta} - \frac{1}{2} \delta^{\alpha}{}_{\beta} R = \frac{8\pi G}{c^{4}} T^{\alpha}{}_{\beta}$$

Matter tells space-time how to curve: field eqs.

$$\nabla_{\alpha} G^{\alpha}{}_{\beta} \equiv 0 \Rightarrow \nabla_{\alpha} T^{\alpha}{}_{\beta} = 0$$

Space-time tells matter how to move: eqs. of motion

PPN metric tensor for a spherical body

$$
\begin{aligned}
g_{00} &= -1 + 2\alpha\frac{U}{c^2} - 2\beta\frac{U^2}{c^4} \\
g_i := g_{0i} &= 4\mu\frac{(\boldsymbol{J}\times\boldsymbol{r})_i}{c^3 r^3} \\
g_{ij} &= (1+2\gamma)\frac{U}{c^2}
\end{aligned}
$$

PPN parameters

Conventional tests of the metric tensor

Standard tests

Test	Experiment	Parameter
Gravitational redshift	GP-A	$\lvert\alpha_{\mathrm{H-maser}} - 1\rvert \leq 1.4\cdot 10^{-4}$
Perihelion shift	Astrophys. observation	$\left\lvert\frac{2(\alpha+\gamma)-\beta}{3} - 1\right\rvert \leq 10^{-4}$
Light deflection	VLBI	$\lvert\gamma - 1\rvert \leq 10^{-4}$
Gravitational time delay	Cassini	$\lvert\gamma - 1\rvert \leq 2\cdot 10^{-5}$?
Lense–Thirring	LAGEOS	$\lvert\mu - 1\rvert \sim 10\%$
Schiff	GP-B	$\lvert\mu - 1\rvert \sim$ 5-10% (expected)

EIH equations of motion

$$\vec{a}_i = \underbrace{\vec{g}_i}_{\text{the Newtonian gravity force}}$$

$$\underbrace{-\sum_{j\neq i}\left(\sum_{k\neq i}\frac{\mu_k}{r_{ik}}+\sum_{k\neq j}\frac{\mu_k}{r_{jk}}\right)\vec{g}_{ij}}_{\text{non-linearity of the gravity field}} + \underbrace{4\sum_{j\neq i}\vec{v}_i\times\left(\vec{v}_j\times\vec{g}_{ij}\right)}_{\text{"gravitomagnetic-like" force}} +$$

$$\underbrace{+\frac{1}{2}\sum_{j\neq i}\left[3v_i^2+4v_j^2-3\left(\vec{v}_i\cdot\hat{r}_{ij}\right)^2\vec{g}_{ij}-6\left(\vec{g}_{ij}\cdot\vec{v}_j\right)\left(\vec{v}_j-\vec{v}_i\right)+\left(\vec{g}_{ij}\cdot\vec{v}_i\right)\vec{v}_i\right]}_{\text{special-relativistic corrections to the gravity force}}$$

$$+\underbrace{\frac{1}{2}\sum_{j\neq i}\frac{\mu_j}{r_{ij}}\left[7\vec{a}_j+\left(\vec{a}_j\cdot\vec{r}_{ij}\right)\vec{r}_{ij}\right]}_{\text{an inductive acceleration-dependent gravity force}} - \underbrace{\left[\frac{1}{2}v_i^2\vec{a}_i+\left(\vec{a}_i\cdot\vec{v}_i\right)\vec{v}_i+3\sum_{j\neq i}\frac{\mu_j}{r_{ij}}\vec{a}_i\right]}_{\text{the post-Newtonian modification of } E=mc^2}$$

PPN equations of motion of extended bodies

$$
\vec{a}_i = \underbrace{\left[1+\frac{\dot{G}}{G}(t-t_0)\right]}_{\text{time-dependent G}} \underbrace{\left(1-\eta\prod\nolimits_i\right)}_{\text{violation of SEP}} \underbrace{\vec{g}_i}_{\text{the Newtonian gravity force}}
$$

a “gravitomagnetic-field” parameter introduced by Soffel et al. (PRD 2008)

$$
\underbrace{-(2\beta-1)\sum_{j\neq i}\left(\sum_{k\neq i}\frac{\mu_k}{r_{ik}}+\sum_{k\neq j}\frac{\mu_k}{r_{jk}}\right)\vec{g}_{ij}}_{\text{non-linearity of the gravity field}} + \underbrace{\left(2\gamma+2-\eta_G\right)\sum_{j\neq i}\vec{v}_i\times\left(\vec{v}_j\times\vec{g}_{ij}\right)}_{\text{"gravitomagnetic-like" force}} +
$$

$$
\underbrace{+\frac{1}{2}\sum_{j\neq i}\left[(2\gamma+1)v_i^2+(2\gamma+2)v_j^2-3\left(\vec{v}_i\cdot\hat{r}_{ij}\right)^2\vec{g}_{ij}-(4\gamma+2)\left(\vec{g}_{ij}\cdot\vec{v}_j\right)\left(\vec{v}_j-\vec{v}_i\right)+\left(\vec{g}_{ij}\cdot\vec{v}_i\right)\vec{v}_i\right]}_{\text{Lorentz-invariance of the gravity force (preferred frame effects)}}
$$

$$
\underbrace{+\frac{1}{2}\sum_{j\neq i}\frac{\mu_j}{r_{ij}}\left[(4\gamma+3)\vec{a}_j+\left(\vec{a}_j\cdot\vec{r}_{ij}\right)\vec{r}_{ij}\right]}_{\text{an inductive acceleration-dependent gravity force}} - \underbrace{\left[\frac{1}{2}v_i^2\vec{a}_i+\left(\vec{a}_i\cdot\vec{v}_i\right)\vec{v}_i+(2\gamma+1)\sum_{j\neq i}\frac{\mu_j}{r_{ij}}\vec{a}_i\right]}_{\text{the post-Newtonian modification of } E=mc^2}
$$

Solution of these equations must be substituted to the solution of equation of a laser pulse propagation (time-delay equation). The PPN time-delay equation has many terms being identical to those in the PPN equations of motion of extended bodies.

‘Conventional’ PPN ranging model

- Any coordinate reference system can be used in relativity to interpret the data.

  True, but making use of inappropriate coordinates easily leads to misinterpretation of gravitational physics.

- Modern computer technology is highly advanced. Data processing can be done in any coordinates irrespectively of the complexity of the equations of motion.

  True, but making use of inappropriate coordinates

  mixes up the spurious, gauge-dependent effects with real physical effects and makes them entangled. There is no unambiguous way to clearly separate gravitational physics from coordinate effects.

- Any post-Newtonian term in the PPN equations of motion has physical meaning and, in principle, can be measured.

  Not true. The PPN equations of motion of the Moon have an enormous number of spurious, gauge-dependent terms that have no physical meaning.

The Gauge Freedom

The gauge condition is imposed on the metric tensor. It simplifies the gravity field equations making their solution mathematically simpler. However, the residual gauge freedom remains. It is defined by the gauge functions ξ^{α} , which obey certain equations and introduce a number of spurious (unphysical) terms to the metric tensor (= gravity field potentials)

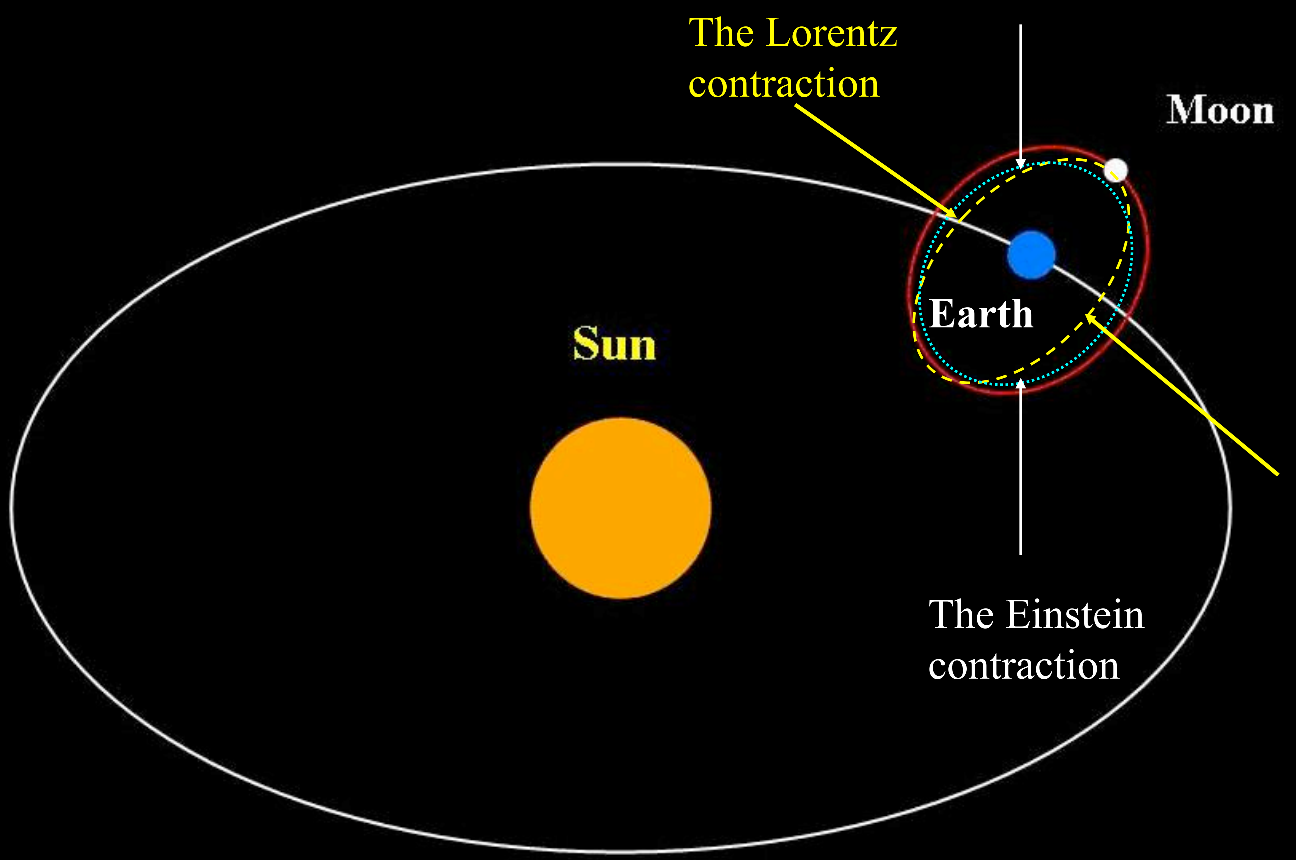
$$\underbrace{w^{\alpha}}_{\text{'new' coordinates}} = \underbrace{x^{\alpha}}_{\text{'old' coordinates}} + \underbrace{\xi^{\alpha}}_{\text{the gauge functions}}(x)$$

$$g_{\alpha\beta}(x) = g_{\mu\nu}(w)\frac{\partial w^{\mu}}{\partial x^{\alpha}}\frac{\partial w^{\nu}}{\partial x^{\beta}} = g_{\alpha\beta}(w) + \xi_{\alpha,\beta} + \xi_{\alpha,\beta} + O(\xi^{2})$$

The spurious terms enter relativistic equations of motion of both the bodies and photons. They must be carefully disentangled from the real physical effects exisiting in the motion of the celestial bodies. The Moon-Earth-Sun system admits a large number of the gauge degrees of freedom, which can be eliminated after transformation to the local inertail frame of the EM barycenter.

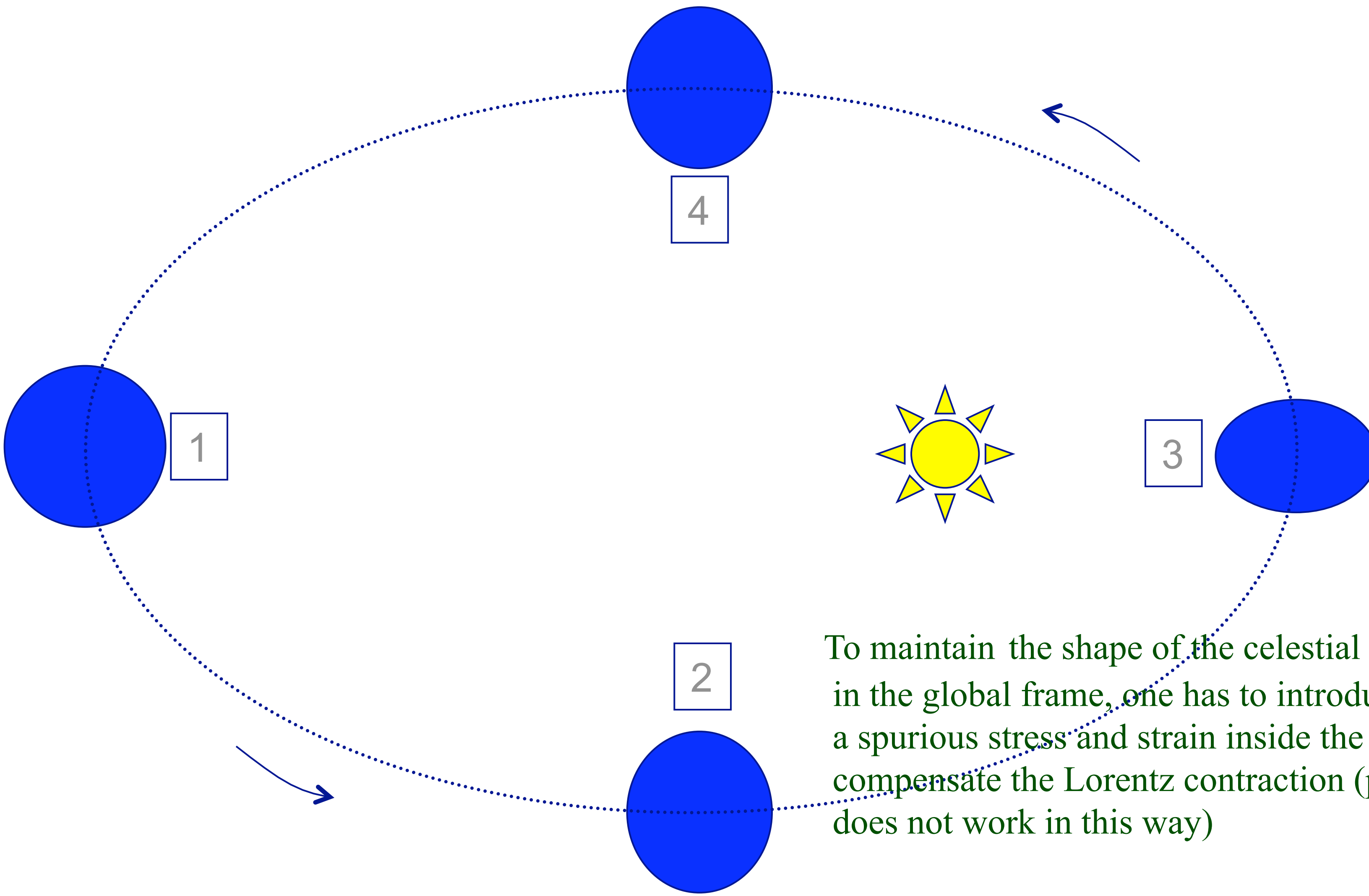
Lorentz and Einstein contractions as the gauge modes

Magnitude of the contractions is about 1 meter!
Ellipticity of the Earth's orbit leads to their annual oscillation of about 2 millimeters. Are they observable by means of LLR?



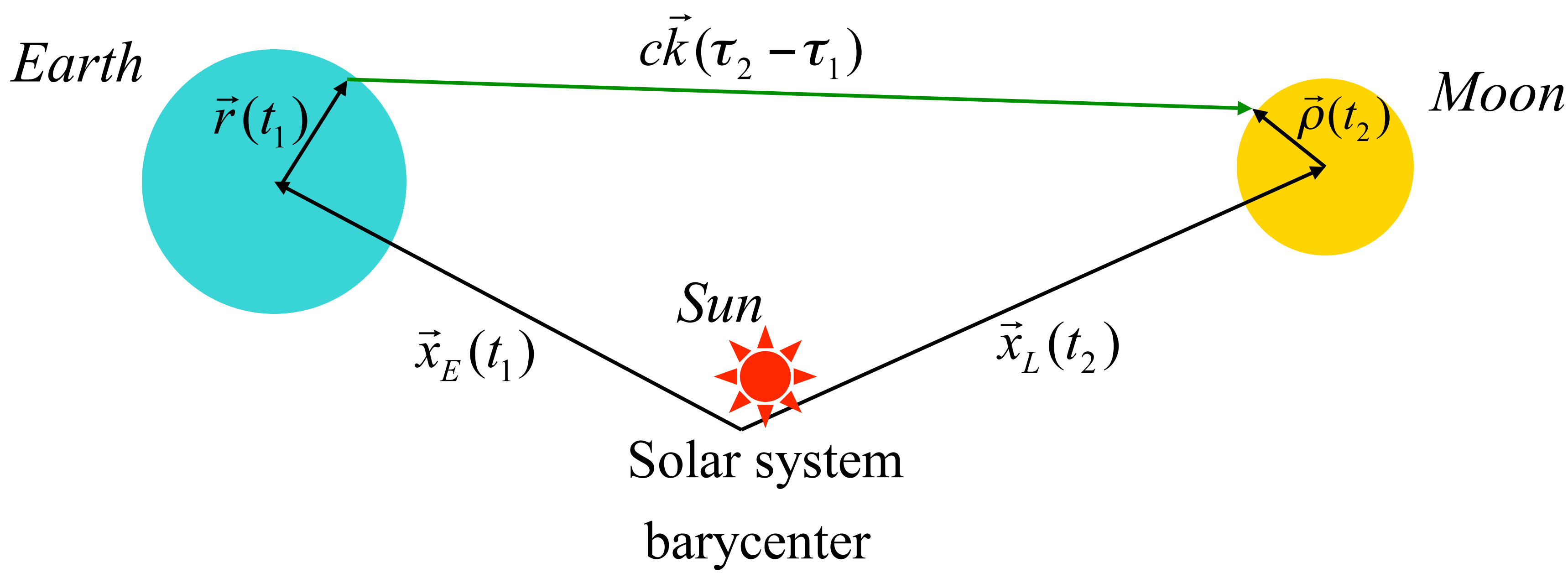
Shape of a moving body in the global frame

Shape of a moving body can be defined in the global frame but it faces major difficulties because of the Lorentz contraction and other (non-linear) frame-dependent coordinate effects. One needs a local frame to work out a such definition.



To maintain the shape of the celestial body in the global frame, one has to introduce a spurious stress and strain inside the body to compensate the Lorentz contraction (physics does not work in this way)

Ranging model of a gauge-invariant theory of gravity



$$\vec{x}_L(t_2)-\vec{x}_E(t_1)=\boxed{\text{Newtonian orbit}}+\boxed{\text{Gauge-dependent terms}}+\boxed{\text{Physical PN perturbations}}$$

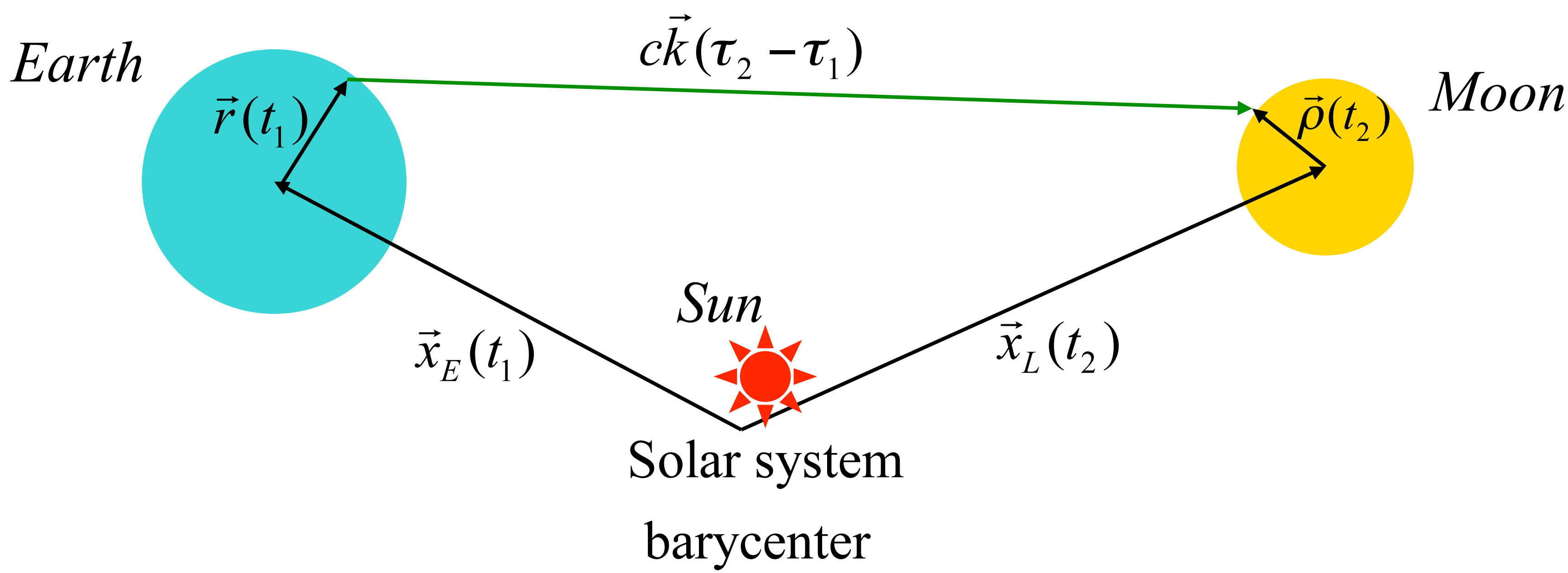
$$\vec{r}(t_1)=\boxed{\text{Newtonian ERP}}+\boxed{\text{Gauge-dependent terms}}+\boxed{\text{Physical PN perturbations}}$$

$$\vec{\rho}(t_2)=\boxed{\text{Newtonian LRP}}+\boxed{\text{Gauge-dependent terms}}+\boxed{\text{Physical PN perturbations}}$$

$$\underbrace{c(\tau_2-\tau_1)}_{\text{Gauge-independent observable time delay}}=\underbrace{\underbrace{\left|\vec{x}_L(t_2)-\vec{x}_E(t_1)+\vec{\rho}(t_2)-\vec{r}(t_1)\right|}_{\text{contains the gauge-dependent terms}}+\underbrace{\boxed{\text{PN time delay (Sun)}}}_{\text{contains the gauge-dependent terms}}+\boxed{\text{PN time delay (Earth)}}}_{\text{all together these terms are gauge-independent}}$$

More details in: Brumberg & Kopeikin, Nuovo Cimento B, ***103***, 63 (1989)

What is happening in the ‘conventional’ PPN ranging model?



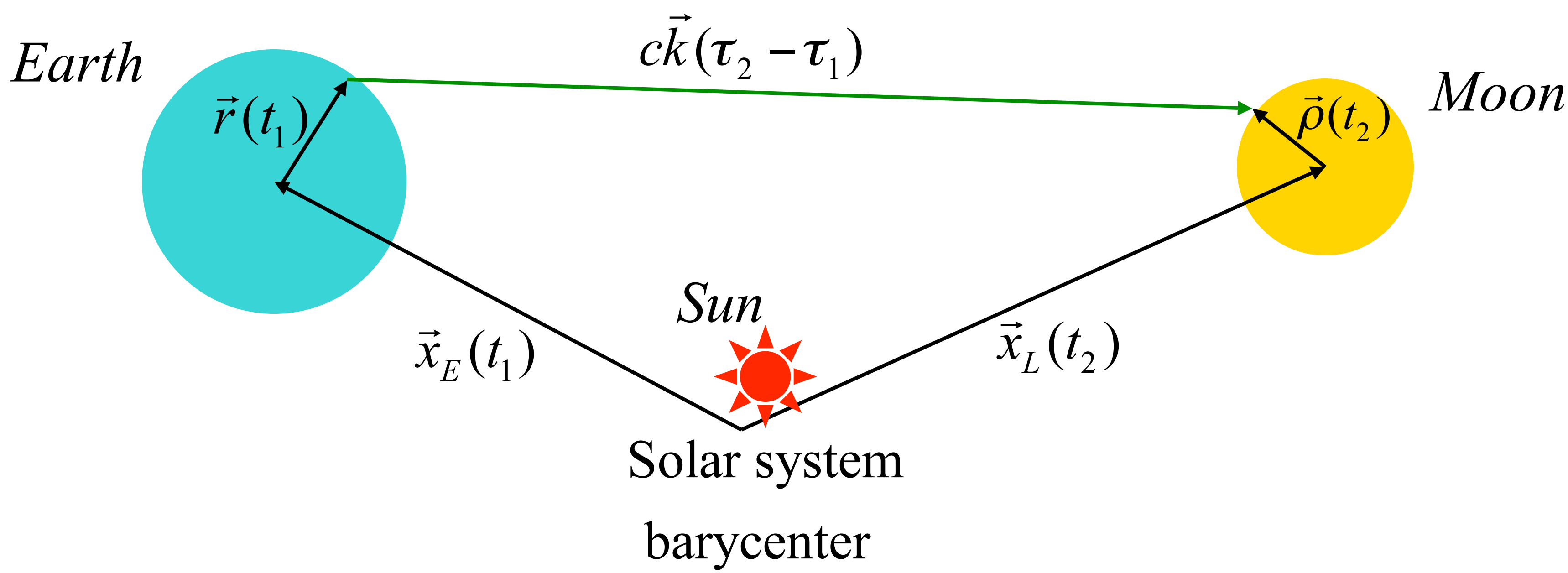
$$\vec{x}_L(t_2)-\vec{x}_E(t_1) = \boxed{\text{Newtonian orbit}} + {\color{red}\eta_G}\boxed{\text{Gauge-dependent terms}} + \boxed{\text{Physical PN perturbations}}$$

$$\vec{r}(t_1) = \boxed{\text{Newtonian ERP}} + \boxed{\text{Gauge-dependent terms}} + \boxed{\text{Physical PN perturbations}}$$

$$\vec{\rho}(t_2) = \boxed{\text{Newtonian LRP}} + \boxed{\text{Gauge-dependent terms}} + \boxed{\text{Physical PN perturbations}}$$

$$\underbrace{c(\tau_2-\tau_1)}_{\substack{\text{Gauge-independent}\\ \text{observable time delay}}} = \underbrace{\underbrace{\left|\vec{x}_L(t_2)-\vec{x}_E(t_1)+\vec{\rho}(t_2)-\vec{r}(t_1)\right|}_{\text{contains the gauge-dependent terms}} + \underbrace{\boxed{\text{PN time delay (Sun)}}}_{\text{contains the gauge-dependent terms}}}_{\text{all together these terms are NOT gauge-independent but proportional to } (\eta_G-1)} + \boxed{\text{PN time delay (Earth)}}$$

Correcting the PPN ranging model



$$\vec{x}_L(t_2) - \vec{x}_E(t_1) = \boxed{\text{Newtonian orbit}} + \eta_G \boxed{\text{Gauge-dependent terms}} + \boxed{\text{Physical PN perturbations}}$$

$$\vec{r}(t_1) = \boxed{\text{Newtonian ERP}} + \eta_G \boxed{\text{Gauge-dependent terms}} + \boxed{\text{Physical PN perturbations}}$$

$$\vec{\rho}(t_2) = \boxed{\text{Newtonian LRP}} + \eta_G \boxed{\text{Gauge-dependent terms}} + \boxed{\text{Physical PN perturbations}}$$

$$\underbrace{c(\tau_2 - \tau_1)}_{\text{Gauge-independent observable time delay}} = \underbrace{\underbrace{\left|\vec{x}_L(t_2) - \vec{x}_E(t_1) + \vec{\rho}(t_2) - \vec{r}(t_1)\right| + \eta_G}_{\text{contains the gauge-dependent terms}} \underbrace{\boxed{\text{PN time delay (Sun)}}}_{\text{contains the gauge-dependent terms}} + \boxed{\text{PN time delay (Earth)}}}_{\text{all together these terms are gauge-independent that is does NOT depend on the parameter } \eta_G}$$

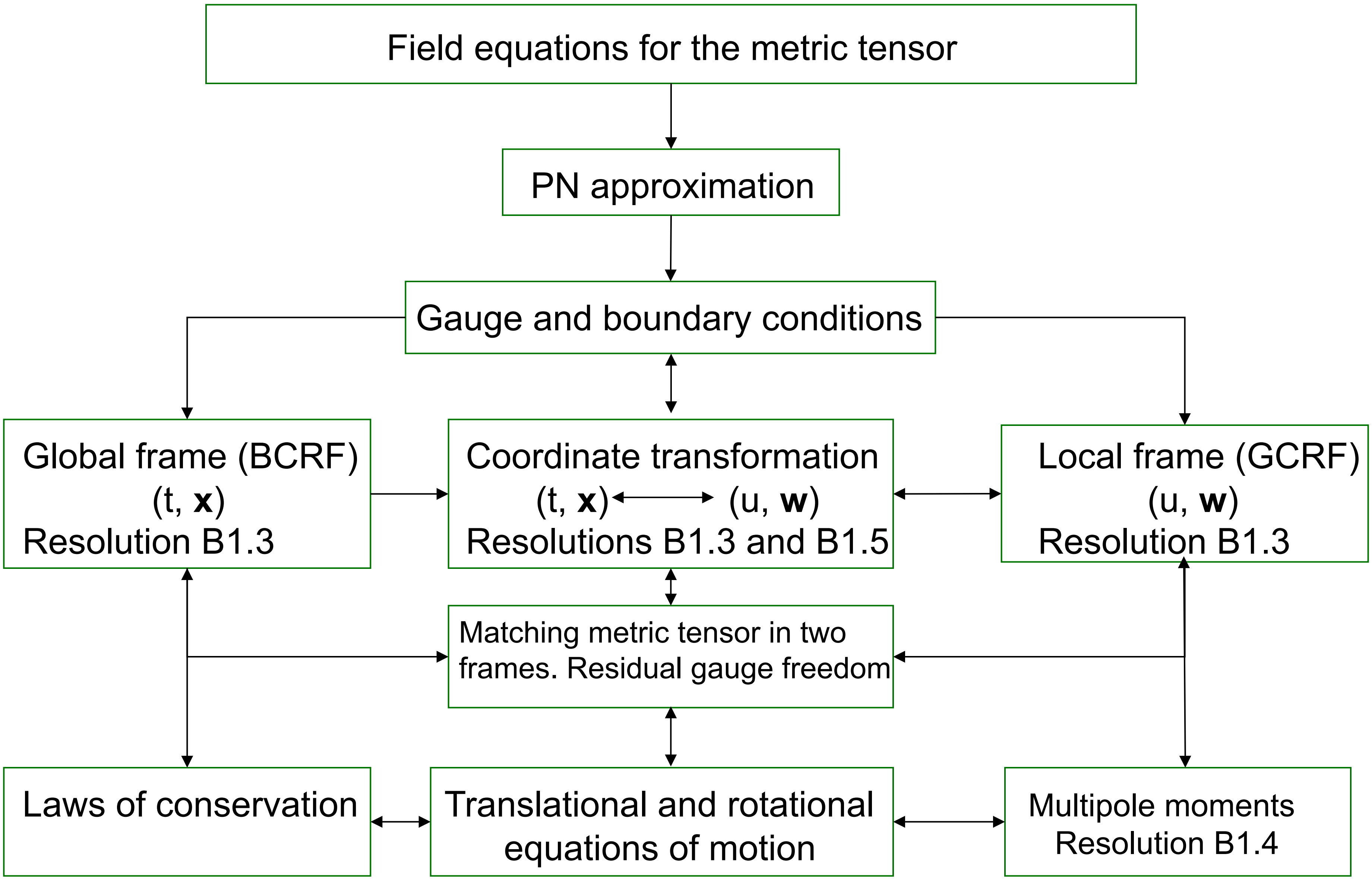
Some details in: Kopeikin & Vlasov, Physics Reports, 2004

Magnitude of the synodic relativistic terms in the radial coordinate of the Moon

Schwarschild	$\boxed{\dfrac{GM_\oplus}{c^2}}$	1 cm
Lense-Thirring	$\boxed{\dfrac{\omega_\oplus R_\oplus}{c}\dfrac{v}{c}R_\oplus}$	0.3 mm
PN Quadrupole	$\boxed{\dfrac{GM_\oplus}{c^2}\left(\dfrac{R_\oplus}{r}\right)^2 J_{2\oplus}}$	2×10^{-4} mm
Gauge-dependent terms	$\boxed{\dfrac{v}{c}\dfrac{v_\oplus}{c}r+\ldots}$	from a few meters down to a few mm
PN Gravitomagnetic	$\boxed{\left(\dfrac{n_\odot}{n_\oplus}\right)^2\dfrac{v}{c}\dfrac{v_\oplus}{c}r}$	a few mm
PN Gravitoelectric	$\boxed{\left(\dfrac{n_\odot}{n_\oplus}\right)^2\left(\dfrac{v_\oplus}{c}\right)^2 r}$	a few cm
Non-linearity of gravity	$\boxed{\left(\dfrac{n_\odot}{n_\oplus}\right)^2\dfrac{GM_\oplus}{c^2}}$	0.1 mm

Gauge-invariant theory of reference frames – IAU 2000

(Brumberg & Kopeikin 1988; Damour, Soffel & Xu 1989)



The gravitomagnetic influence on Earth-orbiting spacecrafts and on the lunar orbit

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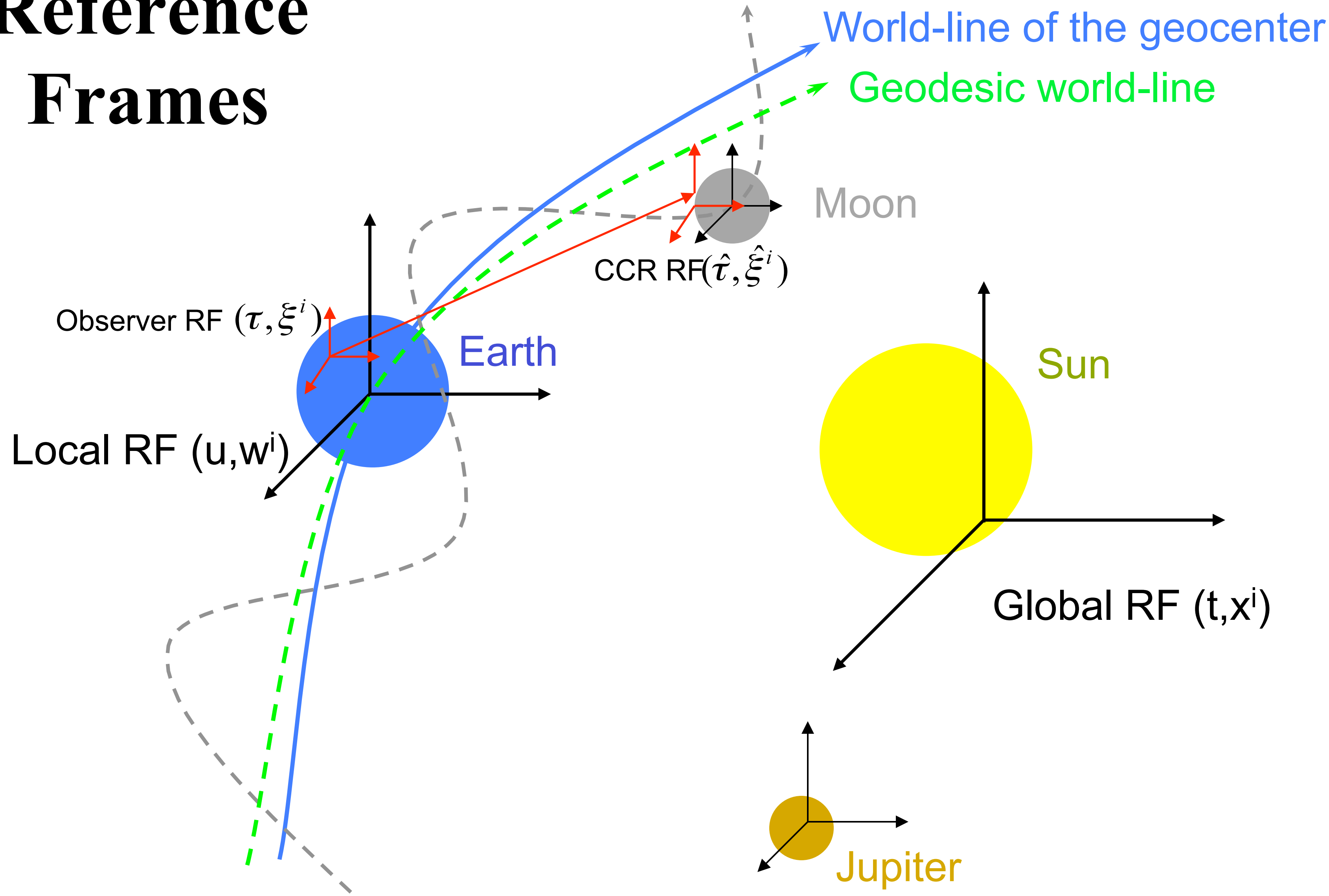
Gravitomagnetic field is covariantly split in the *intrinsic* and *extrinsic* parts, which are generated by rotational and translational currents of matter respectively. The *intrinsic* component has been recently discovered in the LAGEOS spacecraft experiment. We discuss the method of detection of the *extrinsic* tidal component with the lunar laser ranging (LLR) technique. Analysis of the gauge residual freedom in the relativistic theory of three-body problem demonstrates that LLR is currently not capable to detect the *extrinsic* gravitomagnetic effects which are at the ranging level of few millimeters. Its detection requires further advances in the LLR technique that are coming in the next 5-10 years.

PACS numbers: 04.20.-q, 04.80.Cc, 96.25.De

Submitted to the book in memory of J.A. Wheeler.
Editor: I. Ciufolini (2009)

Reference Frames



$$\tau = \tau(u,\vec{w}) \qquad \xi^i = \xi^i(u,\vec{w})$$

$$u = u(t,\vec{x}) \qquad w^i = w^i(t,\vec{x})$$

Lunar theory in the local-inertial frame.

- Earth-Moon system being considered locally, is a binary system on a curved space-time background (Sun, planets).
- Equations of motion of the Earth-Moon system are those of the deviation of geodesics perturbed by the mutual gravitational interaction between Earth and Moon.
- There is a considerable similarity between this problem and that of the evolution of the cosmological perturbations in expanding universe.
- Earth-Moon equations of motion have enormous gauge freedom leading to spurious gauge-dependent modes in motion of the celestial bodies participating in three-body problem.
- The main goal of the advanced lunar theory is
  – to remove all gauge modes,
  – to construct and to match reference frames in the Earth-Moon system with a sub-millimeter tolerance,
  – to ensure that ‘observed’ geophysical parameters and processes are real.
- This is not trivial mathematical problem that requires a peer attention of experts in relativity!

Relativistic mass, center-of-mass and the Earth/ Moon figure

- Definition of mass, center of mass and other multipoles must include the post-Newtonian corrections
- Definition of the body's local reference frame
- Definition of figure in terms of distribution of intrinsic quantities: density, energy, stresses
- Relativistic definition of the equipotential surface – geoid/celenoid (Kopeikin S., 1991, Manuscripta Geodetica, **16**, 301)

Rotation of the Earth/Moon in the Local Frame

(Kopeikin & Vlasov, Physics Reports, 2004)

- Define the intrinsic angular momentum $\boldsymbol{S} = \boldsymbol{I} \cdot \boldsymbol{\Omega}$ of the rotating body in the locally-inertial frame of the body
- Derive equations of the rotational motion in the locally-inertial frame of the body

$$\frac{\mathrm{d}\vec{S}}{\mathrm{d}\tau} = (\text{body's quadrupole}) \times (\text{tidal octupole of the Sun, Earth and planets}) + \ldots$$

$$+ \frac{1}{c^2}\left(\text{post-Newtonian relativistic torque}\right) + \frac{1}{c^4}(\text{neglectibly small})$$

This is the last slide.

THANK YOU!