Refining near-earth object characteristics using numerical synthesis and optimisation

Distinct constituent equations through energy and momentum invariance

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Introduction

A multivariate optimisation technique for refining the physical characteristics of near-earth orbiting objects is being developed. The technique is similar to other types of engineering analysis (Figure 1) in that it is solved through linear algebraic methods. The technique can also be thought of as orbit determination using more object characteristic parameters than are typically considered. Future extensions will possibly model non-spherical geometric shape in relation to drag and solar radiation pressure, and non-natural force production.

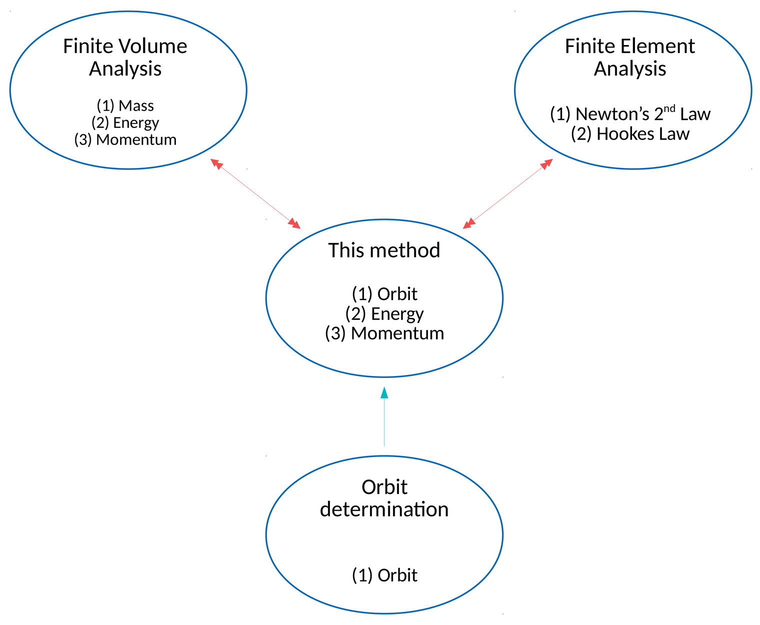

Figure 1: Similar engineering analyses to this method

At the present stage of development, the constituent equations are predominantly derived from the positional continuity of orbital state propagations and are not fully determined. Convergence to a theoretically valid solution occurs, however the solution is not unique and varies with infinitesimal changes to the partial derivatives; a consequence of a non-linear system with limited equation diversity. Constituent equation diversity can be added to the system through energy and momentum invariance in an identical manner to multi-body celestial mechanics problems. The opportunities and challenges of this approach are examined here.

Work to date

Successful convergence to refined values of drag and spherical solar radiation pressure coefficients has been achieved using energy and momentum continuity but the solutions are not unique. Strategies to restrict the scope of solutions returned are being investigated.

The analysis currently uses three constituent equation types[1] for developing the system of equations; orbital continuity (1), energy continuity (2), and momentum continuity (3).

$$\ddot{\mathbf{r}} = \mathbf{F} - GM_{\oplus}\frac{\mathbf{r}}{r^3} \tag{1}$$

$$\frac{v_1^2}{2} - \frac{GM_{\oplus}}{r_1} + \int_{t_1}^{t_2} \mathbf{F}\cdot\frac{d\mathbf{v}}{dt}\,d\mathbf{t} = \frac{v_2^2}{2} - \frac{GM_{\oplus}}{r_2} \tag{2}$$

$$\mathbf{r}_1\times\mathbf{v}_1 + \int_{t_1}^{t_2} \mathbf{r}\times\mathbf{F}\,dt = \mathbf{r}_2\times\mathbf{v}_2 \tag{3}$$

where $\mathbf{r}$ and $\mathbf{v}$ are in the geocentric coordinate system, $\mathbf{F}$ is the perturbative force per unit mass (acceleration) and $M_{\oplus}$ denotes Earth mass. $\mathbf{F}$ can contain spherical harmonic gravity, third-body, drag and solar radiation pressure components.

The disadvantage of this approach is that the three equations use specific (per unit mass) quantities and since (2) and (3) are mathematically derived from (1), they are not distinct enough to form a well-determined system of equations.

Energy and momentum invariance

The use of energy and momentum invariance equations[2] (4) and (5) in place of (2) and (3) provide an opportunity to diversify the system.

$$\xi = \frac{1}{2}\sum_{n}^{i=1} M_i v_i^2 - G\sum_{i=1}^{n}\sum_{\substack{j=1\\ j>i}}^{n}\frac{M_iM_j}{r_{ji}} + M_{\oslash}\int_{t_1}^{t_2}\mathbf{F}\cdot\frac{d\mathbf{v}}{dt}\,d\mathbf{t} = constant \tag{4}$$

$$h = \sum_{n}^{i=1} M_i\left(\mathbf{r_i}\times\mathbf{v_i}\right) + M_{\oslash}\int_{t_1}^{t_2}\mathbf{r}\times\mathbf{F}\,dt = constant \tag{5}$$

where $M_{\oslash}$ represents object mass and n bodies are present within a sphere of influence. This would typically include the Sun and Moon but can include Jupiter and Venus or even all solar system bodies including asteroids. ξ and h are the total energy and total angular momentum and are constant within the sphere of influence due to the laws of conservation of energy and angular momentum.

The advantages of this approach are:

1. The energy and angular momentum invariants in (4) and (5) are algebraically distinct from (1) and (2) due to multiplication by mass and the permuted energy and momentum exchanges between all bodies, which reduces equation redundancy.
2. The presence of the object mass in (4) and (5) makes this characteristic an independent variable that can now be refined. The independence of mass is supplemented by also being present in the denominator of the area-to-mass ratio used in drag and solar radiation pressure force calculations. This also makes object area an independent and refinable characteristic. These additional refinable characteristics were not possible under the system of equations using (1) and (2).

Challenges

Energy and momentum invariance poses complex implementation challenges:

1. Integrating in a geocentric coordinate origin is inconsistent with invariance calculations, so a barycentric coordinate origin is to be used. The manipulation of celestial reference frames is required but position, velocity and acceleration transformations between celestial, intermediate and terrestrial reference frames carry an additional computational performance penalty.
2. Terrestrial Time (T_{TT}) is to be used as the time standard for integration since this analysis needs to evaluate complex geophysical phenomena such as time-varying earth rotation rate and polar motion, which are functions of T_{TT}. Planetary velocities and accelerations are supplied in Barycentric Dynamical Time (T_{DB}) and must be adjusted by the following equations:

$$\frac{dr}{dT_{TT}} = \frac{dr}{dT_{DB}}\,\frac{dT_{TT}}{dT_{DB}} \tag{6}$$

$$\frac{d^2r}{dT_{TT}^2} = \frac{dr}{dT_{DB}}\left(\frac{dT_{TT}}{dT_{DB}}\right)^2 + \frac{dr}{dT_{DB}}\,\frac{d^2T_{TT}}{dT_{DB}^2} \tag{7}$$

3. Since invariance arises from balanced energy and momentum exchanges between all bodies, a planetary ephemeris cannot simply be evaluated since it did not include the presence of the body in its development. Instead a new planetary ephemeris must be integrated including the small but important object force, energy and momentum contributions. The interactions between bodies must be replicated to the best practical extent. This will produce an ephemeris that both reflects the true positions of solar system bodies for third-body force calculations and most importantly, conserves energy and momentum for invariance purposes.
4. A significant contributor to non-Newtonian multi-body effects is the Einstein-Infeld-Hoffmann equation of motion[3]:

$$\begin{aligned} a_A = {} & \sum_{B\neq A}\frac{GM_B\left(\mathbf{r}_B-\mathbf{r}_A\right)}{r_{AB}^3}\Bigg\{1 - \frac{2\left(\beta+\gamma\right)}{c^2}\sum_{C\neq A}\frac{GM_C}{r_{AC}} - \frac{2\beta-1}{c^2}\sum_{C\neq B}\frac{GM_C}{r_{BC}} \\ & + \gamma\left(\frac{v_A}{c}\right)^2 + \left(1+\gamma\right)\left(\frac{v_B}{c}\right)^2 - \frac{2\left(1+\gamma\right)}{c^2}\mathbf{v}_A\cdot\mathbf{v}_B \\ & - \frac{3}{2c^2}\left[\frac{\left(\mathbf{r}_A-\mathbf{r}_B\right)\cdot\mathbf{v}_B}{\mathbf{r}_{AB}}\right]^2 + \frac{1}{2c^2}\left(\mathbf{r}_B-\mathbf{r}_A\right)\cdot\mathbf{a}_B\Bigg\} \\ & + \frac{1}{c^2}\sum_{B\neq A}\frac{GM_B}{r_{AB}^3}\Big[\left[r_A-r_B\right]\cdot\left[\left(2+2\gamma\right)\mathbf{v}_A-\left(1+2\gamma\right)\mathbf{v}_B\right]\Big]\left(\mathbf{v}_A-\mathbf{v}_B\right) \\ & + \frac{\left(3+4\gamma\right)}{2c^2}\sum_{B\neq A}\frac{GM_B\mathbf{a}_B}{r_{AB}} \end{aligned} \tag{8}$$

   The general relativistic conditions of weak gravitational forces and $v \ll c$ are appropriate in this instance, therefore $\beta = 1$ and $\gamma = 1$.
5. The magnitude of individual body energy and momentum values can be very large; for example $h_{earth} \approx 10^{40} kg\, m^2 s^{-1}$. In order for invariance to be successful, these magnitudes must be numerically represented without truncation affecting their significance. Parts of the current program use IEEE-754 quadruple floating point precision (34 decimal digits), but arbitrary-precision arithmetic will need to be used for energy and momentum calculations. Arbitrary-precision arithmetic carries a large computational performance penalty.

Work in progress

1. The modified coordinate origin and time scale changes have been incorporated. The celestial to intermediate frame transformations are currently being coded. When completed, the continuity method will be tested for the same solution convergence rates as were previously achieved.
2. Work is proceeding to replicate the point mass interactions of solar system bodies as found in contemporary ephemerides. Using baseline Newtonian point mass interactions, the Earth currently exhibits a parabolic positional error growth that is about 60 km after one day. It is expected that the inclusion of (8) will improve this value.
3. Work continues on perfecting the energy and momentum balance of solar system bodies to correctly evaluate invariance and thereafter, to assess the improved diversity of the system of equations and the uniqueness of solutions returned.

References

[1] Philip Matthew Fitzpatrick, *Principles of celestial mechanics*, Academic Press, 1970.

[2] David A. Vallado, *Fundamentals of Astrodynamics and Applications*, Microcosm Press, Fourth Edition, 2013.

[3] William M. Folkner, James G. Williams, Dale H. Boggs, Ryan S. Park, Petr Kuchynka, *The Planetary and Lunar Ephemerides DE430 and DE431*, The Interplanetary Network Progress Report Number 42-196, 2014.

Acknowledgements

1. This PhD thesis is part of SERC Research Program 3 (Space Asset Management) and is supervised by Dr James Bennett.